5.3 Higher Order Taylor Methods.

IVP: $\begin{cases} y' = f(t,y), & a \le t \le b \quad (1.1) \\ y(a) = \alpha & (1.2) \end{cases}$

Euler method: $\begin{cases} w_{i+1} = w_i + h f(t_i, w_i) \\ w_0 = \alpha. \end{cases}$

We can also write it as

$$\begin{cases} \dfrac{w_{i+1} - w_i}{h} = f(t_i, w_i), & i = 0, 1, \dots N-1 \quad (1.3) \\ & (1.4) \\ w_0 = \alpha \end{cases}$$

If we substitute the exact Soln. of (1.1)-(1.2) into (1.3)-(1.4). (Euler's equations).

$$\frac{y_{i+1} - y_i}{h} = f(t_i, y_i) + \text{Error} \; ?$$

Now, using Taylor's thm we have shown

$$y_{i+1} = y_i + h f(t_i, y_i) + \frac{h^2}{2} y''(\xi_i)$$

Then, $$\frac{y_{i+1} - y_i}{h} = f(t_i, y_i) + \frac{h}{2} y''(\xi_i)$$ Local truncation error.

We will say that the "local truncation error" of Euler's method is given by

$$\tau_{i+1}(h) = \frac{y_{i+1} - y_i}{h} - f(t_i, y_i) = \frac{h}{2} y''(\xi_i)$$

$\xi_i \in (t_i, t_{i+1})$

or

$$\boxed{\tau_{i+1}(h) = \mathcal{O}(h)} \qquad (2.1)$$

if $y''(\xi_i)$ is bounded.

Derivation of Higher Order Methods.

Expanding $y(t)$ soln. of (1.1)–(1.2) in terms of its nth Taylor polynomial around $t=t_i$

$$y(t_{i+1}) = y(t_i) + h\, y'(t_i) + \frac{h^2}{2} y''(t_i) + \frac{h^3}{3!} y'''(t_i) + \cdots + \frac{h^n}{n!} y^{(n)}(t_i) + \frac{h^{n+1}}{(n+1)!} y^{(n+1)}(\xi_i) \qquad (2.2)$$

$\xi_i \in (t_i, t_{i+1})$

Since $y'(t) = f(t, y(t))$, then

$$y''(t) = \frac{d}{dt}\left(f(t, y(t))\right) = f'(t, y(t)) = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

$$y'''(t) = f''(t, y(t)), \ldots, \quad y^{(k)}(t) = f^{(k-1)}(t, y(t)).$$

Substitution into (2.2).

$$y(t_{i+1}) = y(t_i) + h f(t_i, y(t_i)) + \frac{h^2}{2} f'(t_i, y(t_i)) + \cdots + \frac{h^n}{n!} f^{(n-1)}(t_i, y(t_i)) + \frac{h^{n+1}}{(n+1)!} f^{(n)}(\xi_i, y(\xi_i))$$

From this expression we can define

<u>Taylor Method of order n</u>

$$W_{i+1} = W_i + h f(t_i, w_i) + \cdots + \frac{h^n}{n!} f^{(n-1)}(t_i, w_i)$$

or
$$\begin{cases} \boxed{W_{i+1} = W_i + h T^{(n)}(t_i, w_i)} \\ W_0 = \alpha \end{cases}$$

where
$$\boxed{T^n(t_i, w_i) = f(t_i, w_i) + \frac{h}{2} f'(t_i, w_i) + \cdots + \frac{h^{n-1}}{n!} f^{(n-1)}(t_i, w_i)}$$

Since $f'(t, y(t)) = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial y} y'(t) = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial y} f$

$$\boxed{T^n(t_i, w_i) = f(t_i, w_i) + \frac{h}{2}\frac{\partial f}{\partial t} + \frac{h}{2}\left(f \frac{\partial f}{\partial y}\right)(t_i, w_i) + \cdots + \frac{h^{n+1}}{n!} f^{(n-1)}(t_i, w_i)}$$

Local Truncation error

$$\boxed{T_{i+1}(h) = \frac{h^n}{(n+1)!} f^{(n)}(\xi_i, y(\xi_i))}$$

Show details $\underline{\underline{n=2}}$

Example:
$$\begin{cases} y'(t) = 1 - t + 4y \\ y(0) = 1 \end{cases}$$

Exact Soln: $y(t) = \frac{1}{4}t - \frac{3}{16} + \frac{19}{16}e^{4t}$

Taylor's Method of Order 2

$$\begin{cases} w_{i+1} = w_i + h f(t_i, w_i) + \frac{h^2}{2} f'(t_i, w_i) \\ w_0 = 1 \end{cases}$$

$$f'(t, y(t)) = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial y} y'(t) = -1 + 4y'(t)$$
$$= -1 + 4(1 - t + 4y)$$
$$= 3 - 4t + 16y$$

Then,

$$\boxed{w_{i+1} = w_i + h(1 - t + 4w_i) + \frac{h^2}{2}(3 - 4t + 16w_i)}$$
$$i = 0, 1, 2, \ldots, N-1$$
$$\boxed{w_0 = 1}$$

Similarly, Taylor's method of $\boxed{\text{Order } 3}$

$$\begin{cases} w_{i+1} = w_i + h f(t_i, w_i) + \frac{h^2}{2} f'(t_i, w_i) + \frac{h^3}{3!} f''(t_i, w_i) \\ w_0 = 1 \end{cases}$$

Now, $f'(t,y) = 3 - 4t + 16y$

then, $f''(t, y(t)) = -4 + 16y'(t) = -4 + 16 f(t,y)$

$$= -4 + 16(1 - t + 4y)$$

$$= 2 - 16t + 64y.$$

and

$$\boxed{w_{i+1} = w_i + h(1 - t + 4w_i) + \frac{h^2}{2}(3 - 4t + 16y) + \frac{h^3}{3!}(2 - 16t + 64y)}$$

Taylor's method of $\boxed{\text{Order } 4}$

$$f'''(t, y(t)) = \frac{d}{dt}\left(2 - 16t + 64y(t)\right) = 16 + 64(\underbrace{1 - t + 4y}_{y'(t)})$$

$$= 48 - 64t + 256y$$

then

$$\boxed{w_{i+1} = w_i + h(1 - t + 4w_i) + \frac{h^2}{2}(3 - 4t + 16y) + \frac{h^3}{3!}(2 - 16 + 64y) + \frac{h^4}{4!}(48 - 64t + 256y).}$$